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OPEN MANIFOLDS, OZSVATH-SZABO INVARIANTS AND EXOTIC $\mathbb{R}^4$'S

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Abstract. We construct an invariant of certain open four-manifolds using the Heegaard Floer theory of Ozsvath and Szabo. We show that there is a manifold X homeomorphic to $\mathbb{R}^4$ for which the invariant is non-trivial, showing that X is an exotic $\mathbb{R}^4$. This is the first invariant that detects exotic $\mathbb{R}^4$'s.

1. Introduction

In this paper, we construct invariants of certain open 4-manifolds using the Heegaard Floer theory of Ozsvath and Szabo, and show that our invariants can detect exotic $\mathbb{R}^4$s. Previous constructions of exotic $\mathbb{R}^4$'s used indirect arguments to establish exoticity.

Given an $(n+1)$-dimensional field theory, a direct limit construction can be used to construct an invariant of open $(n+1)$-dimensional manifolds (which we see in detail later). The subtlety in the case of Ozsvath-Szabo invariants is that they do not give a field theory, but satisfy a more complicated composition law. However if we restrict to a class of cobordisms, which we call *admissible cobordisms*, we do get a field theory. Using this, we construct our invariants.

Recall that the Ozsvath-Szabo invariants of a smooth, oriented 3-manifold M associate homology groups to M equipped with a $Spin^c$ structure t. Further, given a smooth cobordism W between 3-manifolds M_1 and M_2 and a $Spin^c$ structure $\mathfrak{s}$ on W, we get an induced map on the groups associated to the restrictions of $\mathfrak{s}$ to M_1 and M_2. To make this into a field theory, one needs a composition rule for a cobordism W_1 from M_1 to M_2 equipped with a $Spin^c$ structure $\mathfrak{s}_1$ and a cobordism W_2 from M_2 to M_3 equipped with a $Spin^c$ structure $\mathfrak{s}_2$ with $\mathfrak{s}_1|_{M_2} = \mathfrak{s}_2|_{M_2}$. However, such $Spin^c$ structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ do not in general uniquely determine a $Spin^c$ structure on the composition $W = W_1 \coprod_{M_2} W_2$ of W_1 and W_2. We do have a weaker composition law, where we sum over $Spin^c$ structures on W restricting to $\mathfrak{s}_1$ and $\mathfrak{s}_2$.

We now find sufficient conditions under which $\mathfrak{s}_1$ and $\mathfrak{s}_2$ uniquely determine a $Spin^c$ structure $\mathfrak{s}$ on W. The $Spin^c$ structures on a manifold X are a torseur of $H^2(X,\mathbb{Z})$. Consider the Mayer-Vietoris sequence for $W = W_1 \cup W_2$

$$\to H^1(W_1) \oplus H^1(W_2) \to H^1(M_2) \xrightarrow{\delta} H^2(W) \to H^2(W_1) \oplus H^2(W_2) \to H^2(M_2)$$

From this sequence, it follows that, given $\mathfrak{s}_1$ and $\mathfrak{s}_2$ as above, there is a unique $Spin^c$ structure $\mathfrak{s}$ on W which restricts to $\mathfrak{s}_1$ and $\mathfrak{s}_2$ if and only if the coboundary map $\delta : H^1(M_2) \to H^2(W)$ is trivial. This is equivalent to the map induced by

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inclusions $H^1(W_1) \oplus H^1(W_2) \to H^1(M_2)$ being surjective. Motivated by this, we make the following definition.

Definition 1.1. A smooth 4-dimensional cobordism W from M_1 to M_2 is admissible if the map induced by inclusion $H^1(W) \to H^1(M_2)$ is surjective.

We shall see basic properties of such cobordisms in Section 2. We now turn to the corresponding notions for open manifolds. Let X be an open 4-manifold which we assume for simplicity has one end. Let $K_1 \subset K_2 \subset \dots$ be an exhaustion of X by compact manifolds and let $M_i = \partial K_i$. We assume here and henceforth (for all exhaustions) that $K_i \subset int(K_{i+1})$. For $i < j$, let $W_{ij} = K_j - int(K_i)$ be cobordisms from M_i to M_j.

Definition 1.2. The exhaustion $\{K_i\}$ of X is said to be admissible if each cobordism W_{ij}, $i, j \in \mathbb{N}$, $i < j$, is admissible. The manifold X is said to be admissible if it has an admissible exhaustion.

We shall need to consider the appropriate notion of $Spin^c$ structures for the ends of 4-manifolds.

Definition 1.3. An asymptotic $Spin^c$ structure $\mathfrak{s}$ on X is a $Spin^c$ structure on $X - K$ for a compact subset $K \subset X$. Two asymptotic $Spin^c$ structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$, defined on $X - K_1$ and $X - K_2$, are said to be equal if there is a compact set $K_0 \supset K_1, K_2$ with $\mathfrak{s}_1|_{M-K_0} = \mathfrak{s}_2|_{M-K_0}$.

Given an admissible open 4-manifold X and an asymptotic $Spin^c$ structure $\mathfrak{s}$, we can define invariants of X, which we call the *End Floer Homology*, using direct limits. We shall see in Section 3 that an admissible exhaustion gives a directed system.

Theorem 1.4. *There is an invariant $HE(X, \mathfrak{s})$ which is the direct limit of the reduced Heegaard Floer homology groups $HF^+_{red}(M_i, \mathfrak{s}|_{M_i})$ under morphisms induced by the cobordisms W_{ij}. Furthermore this is independent of the admissible exhaustion of X.*

We shall also need a *twisted* version of these invariants. Let $K \subset X$ be a compact set, $\mathfrak{s}$ a $Spin^c$-structure on $X - K$ and ω a 2-form on $X - K$. Then we consider the reduced Floer theory with ω-twisted coefficients (as in [14]). Once more we get a directed system whose limit gives an invariant $\underline{HE}(X, \mathfrak{s})$.

By taking an exhaustion of $\mathbb{R}^4$ by balls, we have the following proposition.

Proposition 1.5. *For the unique asymptotic $Spin^c$ structure $\mathfrak{s}$ on $\mathbb{R}^4$ (and any 2-form ω on $\mathbb{R}^4 - K$ with K compact), we have $\underline{HE}(\mathbb{R}^4, \mathfrak{s}) = 0$.*

Our main result is that there are manifolds homeomorphic to $\mathbb{R}^4$ but with non-vanishing end Floer homology.

Theorem 1.6. *There is a 4-manifold X homeomorphic to $\mathbb{R}^4$ such that there is a compact set $K \subset X$, a $Spin^{\mathbb{C}}$ structure $\mathfrak{s}$ on $X - K$ and a closed 2-form ω on $X - K$ with $\underline{HE}(X, \mathfrak{s}) \neq 0$ with ω-twisted coefficients.*

Thus, X is an exotic $\mathbb{R}^4$. Previous constructions of exotic $\mathbb{R}^4$'s used indirect arguments to show that they are exotic. The *End Floer homology* is the first invariant that detects exotic $\mathbb{R}^4$'s.

2. Admissible cobordisms and admissible ends

We henceforth assume that all our manifolds are smooth and oriented and all cobordisms are compact and 4-dimensional. By $W : M_1 \to M_2$ we mean a smooth cobordism from the closed 3-manifold M_1 to the closed 3-manifold M_2. Given $W_1 : M_1 \to M_2$ and $W_2 : M_2 \to M_3$, $W_2 \circ W_1$ denotes the composition of the cobordisms W_1 and W_2.

In this section we prove some simple results concerning admissible cobordisms and admissible ends.

Lemma 2.1. *Suppose $W_1 : M_1 \to M_2$ and $W_2 : M_2 \to M_3$ are admissible cobordisms, then $W = W_2 \circ W_1$ is admissible.*

Proof. We need to show that the map $H^1(W) \to H^1(M_3)$ induced by inclusion is surjective. This is the composition of maps $H^1(W) \to H^1(W_2)$ and $H^1(W_2) \to H^1(M_3)$ induced by inclusion, with the latter surjective by hypothesis. We shall show that the map $H^1(W) \to H^1(W_2)$ is surjective.

Let $\alpha \in H^1(W_2)$ be a class. Let $i_j : M_2 \to W_j$, $j = 1, 2$, be inclusion maps. Consider the Mayer-Vietoris sequence

$$\cdots \to H^1(W) \to H^1(W_1) \oplus H^1(W_2) \xrightarrow{i_1^* + i_2^*} H^1(M_2) \to \dots$$

By admissibility of W_1, there is a class $\beta \in H^1(W_1)$ with $i_1^*(\beta) = i_2^*(\alpha)$. Hence the image of the class $(-\beta, \alpha) \in H^1(W_1) \oplus H^1(W_2)$ in $H^1(M_2)$ is zero, and so $(-\beta, \alpha)$ is the image of a class $\varphi \in H^1(W)$. In particular α is the image of φ under the map induced by inclusion. $\square$

Lemma 2.2. *Suppose $W_1 : M_1 \to M_2$ and $W_2 : M_2 \to M_3$ are cobordisms with $W = W_2 \circ W_1$ admissible. Then W_2 is admissible.*

Proof. By hypothesis the map $H^1(W) \to H^1(M_3)$ is surjective. This factors through the map $H^1(W_2) \to H^1(M_3)$, which must also be surjective. $\square$

We need criteria for when cobordisms corresponding to attaching handles are admissible.

Lemma 2.3. *Let $M = M_1$ be a 3-manifold, W the cobordism corresponding to a handle addition and M_2 the other boundary components of W. The following hold.*

1. *A product cobordism is admissible.*
2. *The cobordism corresponding to attaching a 1-handle to a closed 3-manifold M is admissible.*
3. *If K is a knot in a closed 3-manifold which represents a primitive, non-torsion element in $H_1(M)$, then the cobordism corresponding to attaching a 2-handle to M along K is admissible.*

Proof. We shall show that the map induced by the inclusion from $H_1(M_2)$ to $H_1(W)$ is an isomorphism in each case. As the map on cohomology is the adjoint of this map, it follows that it is a surjection.

The case of a product cobordism is immediate. In the second case we see that $H_1(M_2) = H_1(W) = H_1(M) \oplus \mathbb{Z}$ with the isomorphism induced by inclusion. In the third case we have $H_1(M) = H \oplus \mathbb{Z}$, with $[K]$ generating the $\mathbb{Z}$ component and H isomorphic to the homology of the 3-manifold obtained by surgery about $K \subset M$. It is easy to see that $H_1(W) = H_1(M_2) = H$. $\square$

Now let X be an open manifold and let $K_1 \subset K_2 \subset \dots$ be an exhaustion of X and M_i and W_{ij} be as before.

Lemma 2.4. *The exhaustion $\{K_i\}$ is admissible if and only if each of the manifolds $K_{j+1} - int(K_j)$ is admissible.*

Proof. Each W_{ij} is the composition of cobordisms $K_{j+1} - int(K_j)$. The result follows by Lemmas 2.1 and 2.2. □

Thus, if X is obtained from a compact manifold K by attaching handles as in Lemma 2.3 then X is admissible. Our examples of exotic $\mathbb{R}^4$s will be of this form.

It is immediate from the definition that for any admissible exhaustion K_i, the exhaustion obtained by passing to a subsequence K_{i_j} is admissible. To show independence of our invariants under exhaustions, we need the following lemma.

Lemma 2.5. *Let $K_1 \subset L_1 \subset K_2 \subset L_2 \dots$ be an exhaustion of X with $K_1 \subset K_2 \subset \dots$ and $L_1 \subset L_2 \subset \dots$ admissible exhaustions. Then the exhaustion $L_1 \subset K_2 \subset L_2 \subset K_3 \dots$ is admissible.*

Proof. It suffices to show that the cobordisms $K_{j+1} - int(L_j)$, $j \geq 1$ and $L_j - int(K_j)$, $j \geq 2$ are admissible. This follows from Lemma 2.2 as the cobordisms $K_{j+1} - int(K_j)$ and $L_{j+1} - int(L_j)$ are admissible and we have $K_{j+1} - int(K_j) = (K_{j+1} - int(L_j)) \circ (L_j - int(K_j))$ and $L_{j+1} - int(L_j) = (L_{j+1} - int(K_j)) \circ (K_j - int(L_j))$. □

3. Invariants for admissible ends

We are now ready to define our invariants for an admissible open 4-manifold X. We shall construct invariants based on reduced Heegaard Floer theory HF^+_{red}. First we recall some facts about Ozsvath-Szabo theory.

Associated to each closed, oriented 3-manifold M and $Spin^c$ structure t on M we have abelian groups $HF^+(M,t)$, $HF^-(M,t)$ and $HF^\infty(M,t)$ that fit in an exact sequence

$$\dots \to HF^-(M,t) \to HF^\infty(M,t) \to HF^+(M,t) \to \dots$$

Further, a cobordism $W : M_1 \to M_2$ with a $Spin^c$ structure $\mathfrak{s}$ on W such that $t_i = s|_{M_i}$ induces homomorphisms $F_{W,\mathfrak{s}}$ on these abelian groups which commute with the maps in the above exact sequence.

The group $HF^+_{red}(M,t)$ is defined as the quotient of $HF^+(M,t)$ by the image of $HF^\infty(M,t)$. This is isomorphic to the kernel $HF^-_{red}(M,t)$ of the map from $HF^-(M,t)$ to $HF^\infty(M,t)$. Further, given a cobordism $W : M_1 \to M_2$ with a $Spin^c$ structure $\mathfrak{s}$ on W such that $t_i = s|_{M_i}$, we get an induced homomorphism on the abelian groups $F_{W,\mathfrak{s}} : HF^+_{red}(M_1,t_1) \to HF^+_{red}(M_2,t_2)$ induced by the corresponding homomorphism on HF^+ as the image of $HF^\infty(M_1,t)$ is contained in $HF^\infty(M_2,t)$. This homomorphism is well defined up to choice of sign. We shall denote the above cobordism with its $Spin^c$ structure by $(W,\mathfrak{s}) : (M_1,t_1) \to (M_2,t_2)$.

Further, if $(W_1,\mathfrak{s}_1) : (M_1,t_1) \to (M_2,t_2)$ and $(W_2,\mathfrak{s}_2) : (M_2,t_2) \to (M_3,t_3)$, with $W = W_2 \circ W_1$, we have the composition formula

$$F_{W_2,\mathfrak{s}_2} \circ F_{W_1,s_1} = \sum_{s|_{W_i} = s_i} \pm F_{W,\mathfrak{s}}$$

We shall consider the special case when W_1 is admissible.

Lemma 3.1. *If W_1 is admissible then there is a unique $Spin^c$ structure $\mathfrak{s}$ on W with $\mathfrak{s}|_{W_i} = s_i$. For this $Spin^c$ structure $F_{W_2,\mathfrak{s}_2} \circ F_{W_1,s_1} = \pm F_{W,\mathfrak{s}}$*

Proof. Recall that $Spin^c$ structures are a torseur of $H^2(\cdot, \mathbb{Z})$. Consider the Mayer-Vietoris sequence for $W = W_1 \cup W_2$

$$\to H^1(W_1) \oplus H^1(W_2) \to H^1(M_2) \xrightarrow{\delta} H^2(W) \to H^2(W_1) \oplus H^2(W_2) \to H^2(M_2)$$

By admissibility the map $H^1(W_1) \oplus H^1(W_2) \to H^1(M_2)$ is a surjection, hence $H^2(W) \to H^2(W_1) \oplus H^2(W_2)$ is an injection. This shows uniqueness of the $Spin^c$ structure. As $\mathfrak{s}_1|_{M_2} = t_2 = s_2|_{M_2}$, existence follows from the same exact sequence.

The second statement follows from the first using the composition formula. □

For an admissible exhaustion, it follows that we get a directed system of abelian groups up to sign. We next see that we can choose signs to get a directed system, and the direct limit of the system does not depend on the choice of signs.

Lemma 3.2. *Assume A_i is a sequence of Abelian groups and maps $f_{ij} : A_i \to A_j$, such that for $i < j < k$, $f_{ik} = \pm f_{jk} \circ f_{ij}$. Then we can choose $g_{ij} = \pm f_{ij}$ such that we get a directed system. Furthermore the limit is independent, up to isomorphism, of the choices.*

Proof. Let $g_{1j} = f_{1j}$. For $i < j$, the composition law $g_{1j} = g_{ij} \circ g_{1i}$ uniquely determines sign of $g_{ij} = \pm f_{ij}$, and such a g_{ij} exists as $f_{1j} = \pm f_{ij} \circ f_{1i}$. It is easy to see that this gives a directed system.

For a different choice the maps g_{1j} are replaced by $g'_{ij} = \epsilon_j g_{1j}$, $\epsilon_j = \pm 1$. We get in general a different directed system, with the groups A_i . However, using the isomorphisms $\epsilon_i : A_i \to A_i$ (i.e., $x \mapsto \epsilon_i \times x$ for $x \in A_i$), we get an isomorphism of directed systems. Hence the limits are isomorphic. □

Definition 3.3. The End Floer homology $HE(X, \mathfrak{s})$ is the direct limit of the directed system constructed above.

Proposition 3.4. *The End Floer homology is independent of the admissible exhaustion chosen.*

Proof. By elementary properties of direct limits, the limit does not change on passing to a subsequence of an exhaustion. Given two admissible exhaustions $K_1 \subset K_2 \subset \dots$ and $L_1 \subset L_2 \subset \dots$, by passing to subsequences we can assume that $K_1 \subset L_1 \subset K_2 \subset L_2 \subset \dots$ for the two exhaustions. By Lemma 2.5 the exhaustion $L_1 \subset K_2 \subset L_2 \subset K_3 \dots$ is admissible. As $L_1 \subset L_2 \subset \dots$ and $K_2 \subset K_3 \subset \dots$ are subsequences of this exhaustion, the direct limits for the exhaustions $K_1 \subset K_2 \subset \dots$ and $L_1 \subset L_2 \subset \dots$ are the same (as they are both isomorphic to the direct limit corresponding to the exhaustion $L_1 \subset K_2 \subset L_2 \subset K_3 \dots$). □

We see that this depends only on the diffeomorphism class of the end of X. More precisely, we have the following.

Proposition 3.5. *Suppose X and Y are admissible smooth 4-manifolds and $K \subset X$ and $L \subset Y$ are compact sets so that there is a diffeomorphism $f : X - K \to Y - L$. Then the End Floer homology groups of X and Y are isomorphic.*

Proof. Consider an admissible exhaustion $K_1 \subset K_2 \subset \dots$ with $K \subset K_1$. We define an exhaustion $L_1 \subset L_2 \subset \dots$ of Y by $L_i = L \cup f(K - K_i)$. The map f induces isomorphisms between the terms of the directed systems corresponding to the two

exhaustions. Thus, the End Floer homology groups, which are the limits of these directed systems, are isomorphic. □

We consider the ω-twisted version of this as in [14]. Let $K \subset X$ be a compact manifold and ω a 2-form on $X - K$. We call such a 2-form ω on $X - K$, for K compact, an asymptotic 2-form. Given two closed 2-forms ω_i, $i = 1, 2$, on the complements $X - K_i$ of smooth compact sets K_i, $1 = 1, 2$, we say that ω_1 and ω_2 are asymptotically cohomologous if, for some compact set K, $K_i \subset K$ for $i = 1, 2$, the restrictions of the forms are cohomologous on $X - K$. We can thus speak of asymptotic cohomology classes of asymptotic 2-forms.

We consider an admissible exhaustion with the first term K_1 satisfying $K \subset K_1$. For this, we can define the twisted groups $\underline{HF}^+_{red}(M_i, t_i)$ and homomorphisms associated to W_{ij} which are well defined up to sign and multiplication by powers of T. For any composition $W = W_2 \circ W_1$ associated with the exhaustion as above, the coboundary map $\delta : H^1(M_2) \to H^2(W)$ is zero. It follows by the composition rule for ω-twisted coefficients that we have a directed system up to multiplication by powers of T and sign. As in Lemma 3.2, we can make choices for the homomorphisms to get a directed system and the direct limit is independent of the choices.

The direct limit is the End Floer homology $\underline{HE}(X, \mathfrak{s})$ with ω-twisted coefficients. The following propositions are ananlogous to Propositions 3.4 and 3.5.

Proposition 3.6. *For an aymptotic 2-form ω, the ω-twisted End Floer homology is independent of the choice of admissible exhaustion.*

Proposition 3.7. *Let X and Y are smooth 4-manifolds with admissible ends and ω_X and ω_Y are asymptotic 2-forms on X and Y. If there are compact sets $K \subset X$ and $L \subset Y$, with ω_X and ω_Y defined on $X - K$ and $Y - L$, and a diffeomorphism $f : X - K \to Y - L$ so that ω_X is asymptotically cohomologous to $f^*(\omega_Y)$, then the End Floer homology with ω_X-twisted coefficients of X is homologous to the End Floer homology with ω_Y twisted coefficients of Y.*

4. Exotic $\mathbb{R}^4$'s

We now construct a manifold X homeomorphic to $\mathbb{R}^4$ with $\underline{HE}(X) \neq 0$. This is done by first constructing a convex symplectic manifold W with one convex boundary component N_0 and one convex end and then gluing a compact manifold Y to W along N_0.

4.1. Construction of X.

Let K be a non-trivial slice knot in S^3 and let N be obtained by 0-frame surgery about K. Then N admits a taut foliation by [8], and hence $N \times [0, 1]$ admits a symplectic structure with both ends convex by [5]. The symplectic structure induces a contact structure ξ on N. We shall construct a symplectic manifold Q with one concave boundary component contactomorphic to (N, ξ) and one convex end. The manifold W is obtained by gluing Q to $N \times [0, 1]$.

Let P be the manifold obtained by attaching a 2-handle H to $N \times \{1\}$ corresponding to the surgery cancelling the 0-frame surgery about K. The manifold P has boundary $S \cup N_0$ with $N_0 = N \times \{0\}$ and S a 3-sphere. Let P_0 be $P - S$. Then P_0 has one boundary component, which is diffeomorphic to N, and one end.

Lemma 4.1. *There is a symplectic manifold Q properly homotopy equivalent to P_0 so that the end of Q is convex and the boundary component identified with N_0 is concave with induced contact structure ξ.*

Proof. We construct Q as a Stein cobordism as in [6]. Firstly, by a theorem of Eliashberg [1] (Lemma 2.2 in [6]), there is a Stein cobordism from (N, ξ) to itself, which is thus a Stein structure on $N \times [0,1]$ with $N \times \{0\}$ a concave boundary component and $N \times \{1\}$ a convex boundary component. We construct the manifold Q by attaching 1-handles and 2-handles starting with the convex boundary component, with the 2-handles attached with framing 1 less than the Thurston-Bennequin framing (we call this Legendrian handle addition). By Eliashberg's characterisation of Stein domains [2] (see also [3] and [9]), Q is Stein.

The 1-handles and 2-handles are attached as in Theorem 3.1 of [9], so that the handle H is replaced by a Stein Casson handle. Specifically, by taking a Legendrian representative of $\kappa = \partial H$, we can perform Legendrian handle addition about κ but with incorrect framing, differing from that of H by an integer k. If we attach a handle to κ with this farming but with k self-plumbings (a so called *kinky handle*), then the self-intersection pairing coincides with that obtained by attaching H. As in [9] (where there is an explicit construction in Figure 22), one can attach 1-handles and Legendrian 2-handles to obtain a Stein manifold diffeomorphic to that obtained by attaching a 2-handle with k self-plumbings to κ so that we have the same intersection pairing as adding H.

Thus, we obtain a Stein cobordism with the same intersection pairing as attaching the handle H, but with non-trivial fundamental group. By a lemma of Casson, we can find a family of curves on the boundary of the attached kinky handle, hence the convex boundary of the Stein cobordism, so that attaching 2-handles to these curves (with appropriate framing) gives the manifold obtained on attaching H. As before, we can instead attach kinky handles to obtain a Stein cobordism.

Iterating this procedure gives a non-compact Stein cobordism Q with one concave boundary component and one convex end, which is diffeomorphic to the manifold obtained by attaching a Casson handle in place of H. As Casson handles are properly homotopy equivalent to the interiors of handles, Q is properly homotopy equivalent to P_0.

□

Let W be the symplectic manifold obtained by gluing $N \times [0,1]$ with its sympectic structure obtained by the Gabai-Eliashberg-Thurston theorem, to the symplectic manifold Q, with $N \times \{1\}$ identified with the (concave) boundary of Q. Observe that W is simply-connected as the Casson handle corresponding to the 2-handle H is attached along the meridian of K, which normally generates $\pi_1(N)$. Also observe that in the proof of Lemma 4.1, following Theorem 3.1 of [9], the handles attached are as in Lemma 2.3, and hence the corresponding exhaustion is admissible.

Next, let Y' be obtained from B^4 by attaching a 2-handle along K with framing 0. Then $\partial Y' = N$. As K is slice, the generator of $H_2(Y) = \mathbb{Z}$ can be represented by an embedded sphere Σ. Let Y be obtained from Y' by performing surgery along Σ. Glue W to Y along $\partial Y = N = N \times \{0\}$ to obtain X.

By a Mayer-Vietoris argument, X has the homology of $\mathbb{R}^4$. Further, as $\pi_1(Y)$ is normally generated by a meridian of K, to which a Casson handle is attached, $\pi_1(X) = 1$. Finally, the end of X is properly homotopic to the end of $P_0 = P - S$, and hence Y is simply-connected at infinity. Thus Y is homeomorphic to $\mathbb{R}^4$ by Freedman's theorem [7].

4.2. **Non-Vanishing of End Floer homology.** Finally, we show that the End Floer homology for X does not vanish. Consider the exhaustion of X with $K_1 = Y$, hence $M_1 = N$ and $K_2, K_3, \dots$ being the level sets after attaching successive handles as above. Note that $X - K_1$ is symplectic with symplectic form ω, and each of the cobordisms W_{1j} is a convex symplectic manifold with two convex boundary components M_1 and M_j. Hence W_{1j} embeds in a symplectic 4-manifold $Z = X_1 \cup W_{1j} \cup X_j$ with both components of $Z - W_{1j}$ having $b_2^+ > 0$ by results of Eliashberg [4] and Kronheimer-Mrowka [10]. Here X_1 and X_j are manifolds with boundaries M_1 and M_j, respectively.

We shall consider ω-twisted coefficients and the $Spin^c$ structure $\mathfrak{s}$ associated to ω. Recall that ω-twisted coefficients are coefficients determined by ω as follows: for a 3-manifold $P \subset M$, we consider $\mathbb{Z}[\mathbb{R}]$ as a module over $\mathbb{Z}[H^1(N,\mathbb{Z})]$ via the ring homomorphism $[\gamma] \mapsto T^{\int_N [\gamma]\wedge\omega}$. Ozsvath and Szabo show that we have induced maps with ω-twisted coefficients satisfying an appropriate composition formula. By an application of Stokes theorem, we deduce the relation

$$\int_N [\gamma] \wedge \omega = \int_Z \delta[\gamma] \wedge \omega$$

Let t_i be the $Spin^c$ structure on M_i induced by $\mathfrak{s}$. We first construct an element $x_1 \in \underline{HF}^+(M_1, t_1)$ whose image $z_1 \in \underline{HF}^+_{red}(M_1, t_1)$ will be shown to have non-zero image in the direct limit giving the End Floer homology.

Let $P \subset X_1$ be an admissible cut in the terminology of Ozsvath and Szabo. Then as $\delta H^1(P) = 0$, ω-twisted coefficients coincide with untwisted coefficients(as $\int_P [\gamma] \wedge \omega = \int_Z \delta[\gamma] \wedge \omega = 0$). Let the closures of the components of $X_1 - P$ be U and V, with $M_1 \subset \partial V$. Let $B_1 \subset U$ be a ball. As in the construction of the closed 4-manifold invariants, we obtain an element $\xi \in \underline{HF}^+(P, \mathfrak{s}) = HF^+(P, \mathfrak{s})$ as the image of the generator of $HF^-(S^3)$ using the isomorphism between HF^-_{red} and HF^+_{red}. We define x_1 to be the image $\underline{F}_V(\xi)$ of ξ in $\underline{HF}^+(M_1, t_1)$ under the map induced by the cobordism V and let z_1 be its image in reduced Floer homology.

Let $x_j \in \underline{HF}^+(M_j, t_j)$ be the image of x_1 under the cobordism induced by W_{1j} and let $z_j \subset \underline{HF}^+_{red}(M_j, t_j)$ be corresponding image of z_1.

Lemma 4.2. *For every $j \geq 0$, $z_j \neq 0$.*

Proof. Let $j > 1$ be fixed. Let $W = W_{1j} \cup X_j$ and let B_2 be a ball in X_j. We shall show that the image of x_1 in $HF^+(S^3, \mathfrak{s}_0)$ under the map induced by $W - B_2$ is non-zero.

Lemma 4.3. *The image $\underline{F}_{W-B_2}(x_1)$ of x_1 in $HF^+(S^3, \mathfrak{s}_0)$ under the map induced by $W - B_2$ is non-zero.*

Proof. Our proof is based on the proof of Theorem 4.2 in [14]. We use the product formula with ω-twisted coefficients

$$\sum_{\eta \in H^1(M_1,\mathbb{Z})} \Phi_{M,\mathfrak{s}+\delta\eta} T^{<\omega \cup c_1(s+\delta\eta),[M]>} = \underline{F}_{W-B_2} \circ \underline{F}_V(\xi) = \underline{F}_{W-B_2}(x_1)$$

Thus it suffices to show that the left hand side does not vanish. By results of Ozsvath and Szabo on the closed four-manifold invariants for symplectic manifolds (as in [14], Theorem 4.2), the lowest order term of the left hand side, which is a polynomial in T, is 1. It follows that $\underline{F}_{W-B_2}(x_1) \neq 0$, completing the proof. $\square$

Now, by Lemma 3.1, as W_{1j} is admissible, this factors through the map induced by W_{1j}, and hence the image of x_j in $HF^+(S^3, \mathfrak{s}_0)$ is non-zero. But as the cobordism $X_j - int(B_2)$ has $b_2^+ > 0$, the induced map on $\underline{HF}^\infty$ is zero. It follows that x_j is not in the image of $\underline{HF}^\infty(M_i, t_i)$, i.e. $z_j \neq 0$, as claimed. $\square$

Thus, the End Floer homology of X does not vanish. We have seen that X is homeomorphic to $\mathbb{R}^4$. This completes the proof of Theorem 1.6. $\square$

References

1. Eliashberg, Ya. M. *Complexification of contact structures on* 3*-dimensional manifolds (Russian)*, Uspekhi Mat. Nauk 40 (1985), 161–162.
2. Eliashberg, Yakov *Topological characterization of Stein manifolds of dimension* > 2, Internat. J. Math. 1 (1990), 29–46.
3. Eliashberg, Yakov *Legendrian and transversal knots in tight contact* 3*-manifolds*, Topological methods in modern mathematics (Stony Brook, NY, 1991), 171–193, Publish or Perish, Houston, TX, 1993.
4. Eliashberg, Yakov *A few remarks about symplectic filling*, Geom. Topol. **8** (2004), 277–293.
5. Eliashberg, Yakov M.; Thurston, William P. *Confoliations*, University Lecture Series, **13**, American Mathematical Society, 1998.
6. Etnyre, John B.; Honda, Ko *On symplectic cobordisms*, Math. Ann. 323 (2002), 31–39.
7. Freedman, Michael Hartley *The topology of four-dimensional manifolds.* J. Differential Geom. **17** (1982), 357–453
8. Gabai, David *Foliations and the topology of* 3*-manifolds. III* J. Differential Geom. **26** (1987), 479–536
9. Gompf, Robert E. *Handlebody construction of Stein surfaces* Ann. of Math. (2) **148** (1998), 619–693.
10. Kronheimer, P. B.; Mrowka, T. S. *Witten's conjecture and property P*, Geom. Topol. **8** (2004), 295–310.
11. Ozsvath, Peter; Szabo, Zoltan *Holomorphic discs and topological invariants for closed three-manifolds*, to appear in the Annals of Mathematics.
12. Ozsvath, Peter; Szabo, Zoltan *Holomorphic triangles and invariants of smooth four-manifolds*, preprint
13. Ozsvath, Peter; Szabo, Zoltan *Holomorphic triangle invariants and the topology of symplectic four-manifolds.* Duke Math. J. 121(2004), no. 1, 1–34
14. Ozsvath, Peter; Szabo, Zoltan *Holomorphic disks and genus bounds*, Geom. Topol. **8** (2004), 311–334

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